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Relativistic orthogonal polynomials are Jacobi polynomials

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Abstract. We identify the recently studied relativistic orthogonal polynomials in terms of Jacobi polynomials, so all their properties follow from the corresponding properties of Jacobi polynomials through a change of variable.

1. Introduction

Recently several authors have studied the so-called relativistic Hermite, Laguerre and Jacobi polynomials [1–10]. They noted that these relativistic polynomials are orthogonal with respect to varying weight functions, meaning that the weight function depends on the degree of the polynomials involved. The relativistic Hermite, Laguerre and Jacobi polynomials were introduced in [1], [8] and [3], respectively. In addition, the so-called relativistic Szegö polynomials were introduced in [4].

The purpose of this paper is to point out that all these relativistic polynomials are Jacobi polynomials in a different variable. In general what happens is the following. Let $\{p_n(x)\}$ be a family of orthogonal polynomials satisfying

$$\int_{-\infty}^{\infty} p_m(x)\, p_n(x)\, \mathrm{d}\mu(x) = h_n \delta_{m,n}. \tag{1.1}$$

Clearly every $q_n(x)$ defined through

$$q_n(x) := (\alpha x + \beta)^n p_n\left(\frac{ax+b}{\alpha x + \beta}\right) \tag{1.2}$$

is a polynomial of degree n if $\alpha = 0$ and $\beta \neq 0$ and also if $\alpha \neq 0$ provided that $p_n(a/ \neq 0$. This is true because the leading term in $q_n(x)$ is $\lim_{x\to\infty} x^{-n} q_n(x)$. Now observe that a change of variable in (1.1) will make the $q_n(x)$'s of (1.2) orthogonal with respect to a varying weight. Explicitly this is the relationship

$$\int_{-\infty}^{\infty} q_m(x) q_n(x) (\alpha x + \beta)^{-m-n}\, \mathrm{d}\mu\left(\frac{ax+b}{\alpha x + \beta}\right) = h_n \delta_{m,n}. \tag{1.3}$$

Although polynomials orthogonal with respect to a varying weight function are interesting, the ones obtained through the above construction do not yield any new information. In fact Nagel [11] has already pointed out that the relativistic Hermite polynomials are ultraspherical polynomials but this has not seemed to stop others from studying them, see for example [12]. We certainly hope this paper will close this chapter on relativistic orthogonal polynomials.

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2. Identification

What is needed here, other than the standard notation for shifted factorials and hypergeometric functions [13], is the following set of elementary identities,

$$(\sigma)_s = \Gamma(\sigma + s)/\Gamma(\sigma) \tag{2.1}$$
$$(2\sigma)_{2s} = 2^{2s}(\sigma)_s(\sigma + 1/2)_s \tag{2.2}$$
$$\frac{n!}{(n-s)!} = (-1)^s(-n)_s \tag{2.3}$$

and the (Pfaff–)Kummer transformation [13]

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) = {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix} \middle| \frac{x}{x-1}\right). \tag{2.4}$$

Also recall that [14]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| \tfrac{1}{2}(1-x)\right). \tag{2.5}$$

In the case of the case of the relativistic Hermite polynomials $H_n^{(N)}(x)$

$$H_n^{(N)}(x) := \sum_{k=0}^{[n/2]} a_{n,n-2k}(2x)^{n-2k} \tag{2.6}$$

where

$$a_{n,n-2k} := \frac{(-1)^k n! N^k \Gamma(N+1/2)\Gamma(n+2N)}{k!(n-2k)!(2N)^n\Gamma(N+k+1/2)\Gamma(2N)} = \frac{(2N)_n(-n)_{2k}(-N)^k}{(N+1/2)_k k!} \tag{2.7}$$

where we have used (2.1) and (2.2). Upon using (2.3) we see that $H_n^{(N)}(x)$ has the hypergeometric representation

$$H_n^{(N)}(x) = \frac{(2N)_n x^n}{N^n}\,{}_2F_1\left(-n/2, -(n-1)/2 \,\middle|\, -N/x^2\right). \tag{2.8}$$

Now the Pfaff–Kummer transformation (2.4) gives

$$\begin{aligned} H_{2n}^{(N)}(x) &= \left(1 + \frac{x^2}{N}\right)^n \frac{(2N)_{2n}}{N^n}\,{}_2F_1\left(\begin{matrix} -n, n+N \\ N+1/2 \end{matrix} \middle| \frac{N}{N+x^2}\right) \\ &= \frac{(2N)_{2n} n!}{N^n(N+1/2)_n}\left(1+\frac{x^2}{N}\right)^n P_n^{(N-1/2,-1/2)}\left(\frac{x^2-N}{x^2+N}\right). \end{aligned} \tag{2.9}$$

Similarly

$$H_{2n+1}^{(N)}(x) = x\,\frac{(2N)_{2n+1} n!}{N^{n+1}(N+1/2)_n}\left(1+\frac{x^2}{N}\right)^n P_n^{(N-1/2,1/2)}\left(\frac{x^2-N}{x^2+N}\right).$$

According to [8] and [3] the relativistic Laguerre polynomials are

$$L_n^{(\alpha,N)}(x) := \sum_{j=0}^{n}\binom{n+\alpha}{n-j}\prod_{k=n-j}^{n-1}\left(1+\frac{2k+1}{2N}\right)\frac{(-x)^j}{j!}. \tag{2.10}$$

It is easy to apply (2.1) and (2.3) and obtain

$$
\begin{aligned}
L_n^{(\alpha,N)}(x) &= \frac{(\alpha+1)_n}{n!}\sum_{j=0}^{n}\frac{(-n)_j(-N-n+1/2)_j}{(\alpha+1)_j\, j!}\left(-\frac{x}{N}\right)^j \\
&= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n, -N-n+1/2\\ \alpha+1\end{matrix}\,\right|-\frac{x}{N}\right) \\
&= \frac{(\alpha+1)_n}{n!}\left(1+\frac{x}{N}\right)^n {}_2F_1\left(\left.\begin{matrix}-n, \alpha+N+n+1/2\\ \alpha+1\end{matrix}\,\right|\frac{x}{x+N}\right)
\end{aligned}
$$

where we have again used the Pfaff–Kummer transformation (2.4). This shows that

$$
L_n^{(\alpha,N)}(x) = \left(1+\frac{x}{N}\right)^n P_n^{(\alpha,N-1/2)}\left(\frac{N-x}{N+x}\right).
$$

The relativistic Jacobi polynomials were introduced in [3] as

$$
P_n^{(\alpha,\beta,N)}(x) := \sum_{k=0}^{n}\binom{n+\alpha}{n-j}\frac{(N-\beta)^k(x-1)^k}{(2N)^k k!}\left(n+\frac{(\alpha+=1)N+/2}{N-\beta}\right)_k. \tag{2.11}
$$

Therefore

$$
P_n^{(\alpha,\beta,N)}(x) = P_n^{(\alpha,\gamma)}(y) \tag{2.12}
$$

with

$$
y := \frac{x(N-\beta)+\beta}{N} \qquad \text{and} \qquad \gamma := \frac{\beta(\alpha+N+3/2)}{N-\beta}. \tag{2.13}
$$

With the identification (2.12) and (2.13) the distribution of zeros and other properties of the relativistic Jacobi polynomials [5, 6] follow from the corresponding results for the Jacobi polynomials.

We finally come to the so-called relativistic Szegö polynomials. In [4] it is claimed that Szegö [14] introduced the generalized Hermite polynomials

$$
H_{2n}^{\mu}(x) = (-1)^n 4^n n! L_n^{(\mu-1/2)}(x^2) \qquad H_{2n+1}^{\mu}(x) = (-1)^n 4^n n!(2x) L_n^{(\mu+1/2)}(x^2). \tag{2.14}
$$

It is not really right to call these Szegö polynomials because Szegö [14] pointed out that these polynomials are Laguerre polynomials in a different normalization. The point is that there is no reason to study them or their zeros separately [8], if we can get the same information from the extensive results in Szegö's book [14]. The relativistic version of the $H_n^{(\mu)}$'s is expressed in terms of the relativistic Laguerre polynomials which we have already identified as Jacobi polynomials. We leave to the reader the exercise of writing them explicitly as Jacobi polynomials.

3. Conclusion

The relativitic polynomials that have been the subject of several recent publications are not new. They are Jacobi polynomials and their properties follow from the properties of Jacobi polynomials.

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